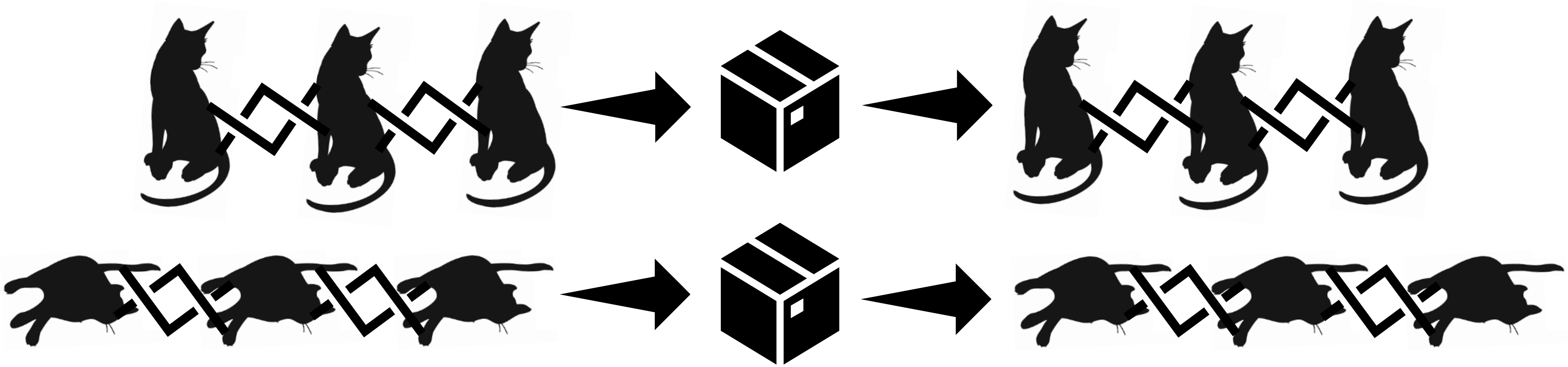
Stabilizer codes

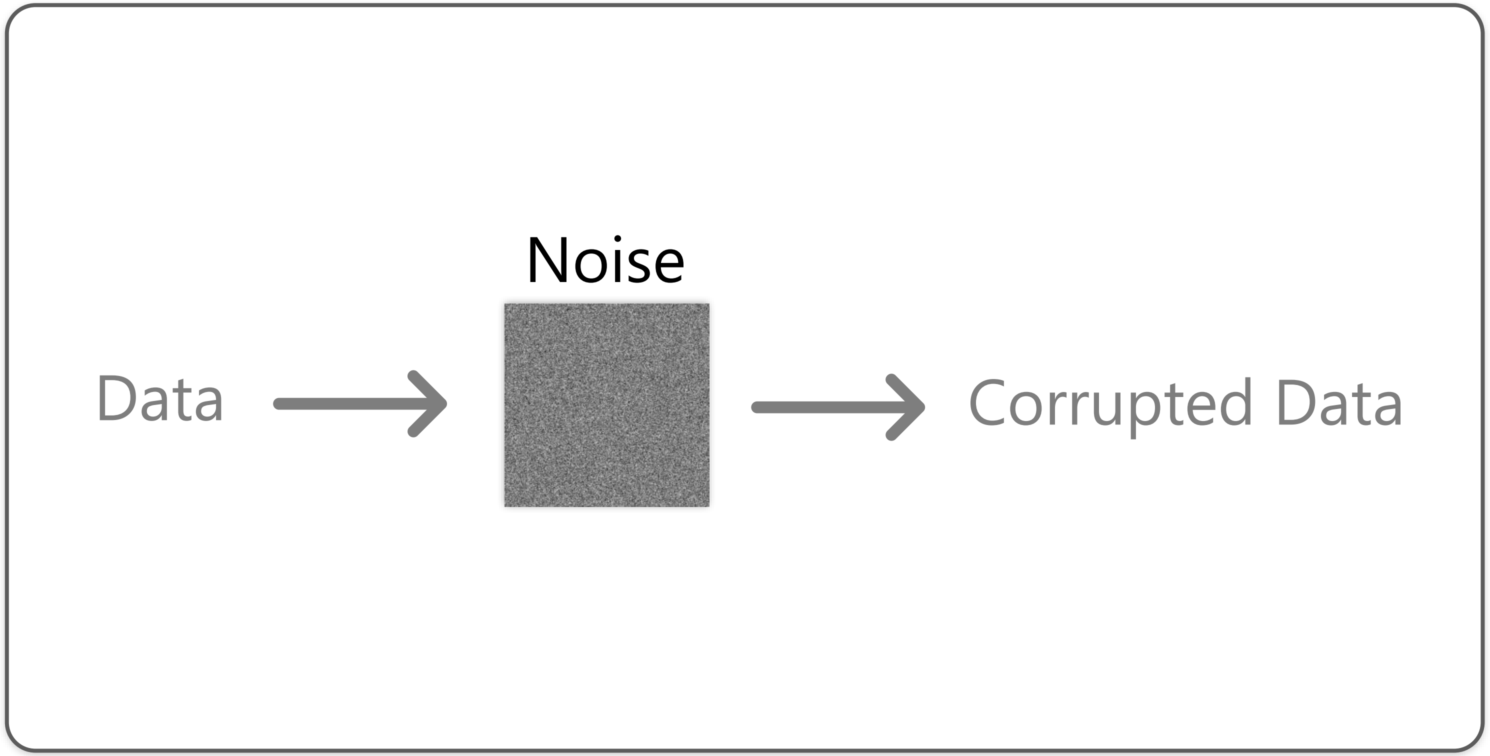
Simão Gonçalves Eusébio

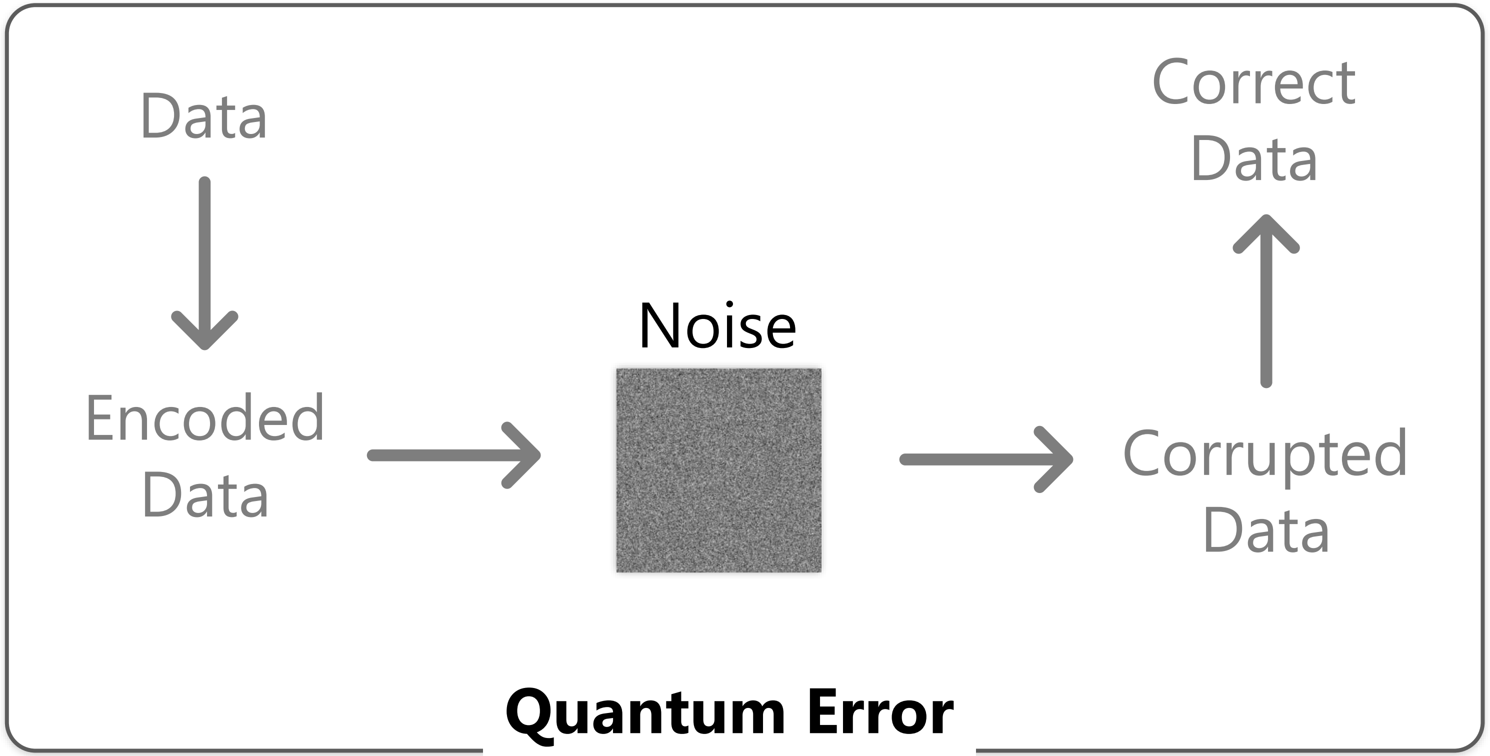


Algebraic and Geometric Methods in Engineering and Physics

February 2021

Outline


Noise
Data
Corrupted Data



Quantum Error Correction

Some elementary notions

Qubit: The elementary unit of information

Classically

Bit

$0 \; 1$

Quantically

Qubit

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle \qquad a, b \in \mathbb{C}$$

$$\boxed{\begin{array}{l} |\psi\rangle \sim e^{i\theta}\,|\psi\rangle \qquad \theta \in \mathbb{R} \\ |a|^2 + |b|^2 = 1 \end{array}}$$

Generalization for a system of N qubits:

$$|\psi\rangle = \sum_{i=0}^{2^N-1} \alpha_i\,|i\rangle\,, \qquad \sum_{i=0}^{2^N-1} |\alpha_i|^2 = 1$$

$$|\psi\rangle = \sum_{i=0}^{2^N-1} \alpha_i\,|i\rangle = \alpha_0\,|0\ldots0\rangle + \alpha_1\,|0\ldots01\rangle + \alpha_2\,|0\ldots010\rangle + \ldots + \alpha_{2^N-1}\,|1\ldots1\rangle$$

$$\langle 0|1\rangle = 0$$
$$\langle 0|0\rangle = \langle 1|1\rangle = 1$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|s\rangle = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \quad \langle s| = (|s\rangle)^\dagger = \begin{bmatrix} s_1^* & s_2^* \end{bmatrix}$$

Tensor product: From 1 to N

$$|v\rangle \otimes |s\rangle = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \otimes \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} v_1 \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} \\ \vdots \\ v_m \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} \end{bmatrix}$$

<u>Dimension</u>: $m \times n$

$$\boxed{|010\rangle = |0\rangle \otimes |1\rangle \otimes |0\rangle}$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, |01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, |10\rangle = |1\rangle \otimes |0\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} |11\rangle = |1\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \dots & A_{1n}B \\ A_{21}B & A_{22}B & \dots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \dots & A_{mn}B \end{bmatrix}$$

$$(A \otimes B)(C \otimes D) = (AC \otimes BD)$$

$$(A \otimes B) |\psi\rangle \otimes |\phi\rangle = (A |\psi\rangle \otimes B |\phi\rangle)$$

$$X = \sigma_1 = \sigma_x \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \sigma_2 = \sigma_y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \sigma_3 = \sigma_z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Eigenvectors:

$$|\psi_{x+}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad |\psi_{y+}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}, \quad |\psi_{z+}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|\psi_{x-}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad |\psi_{y-}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}, \quad |\psi_{z-}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Eigenvalues: ± 1

$$det(\sigma_i) = -1$$
$$tr(\sigma_i) = 0$$
$$\sigma_i^2 = I$$
$$\sigma_i^\dagger = \sigma_i$$
$$\{\sigma_i, \sigma_j\} = 2I\delta_{ij}$$
$$-i\sigma_1\sigma_2\sigma_3 = I$$

$$A = a_0 I + a_1 X + a_2 Y + a_3 Z$$
$$\sigma_0 \equiv I$$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = |+\rangle\langle+| - |-\rangle\langle-|$$

Eigenvectors: $|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$ $\quad |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ **Eigenvalues:** ± 1

$$|z\rangle \overset{H^{\otimes n}}{\rightarrow} \frac{1}{\sqrt{2^n}} \sum_{s=0}^{2^n-1} (-1)^{z \cdot s} |s\rangle$$

$$|-\rangle = H|1\rangle \quad |+\rangle = H|0\rangle$$

$$H = H^\dagger \qquad HH^\dagger = H^2 = I$$

Classical linear codes

$[n, k, d]$ linear code

$G \in Mat_{n\times k}(F_2)$	$H \in Mat_{(n-k)\times n}(F_2)$
$G = \begin{bmatrix} \vert & & \vert \\ g_1 & \dots & g_k \\ \vert & & \vert \end{bmatrix}$	$H = \begin{bmatrix} — h_1 — \\ \vdots \\ — h_{n-k} — \end{bmatrix}$
$x \in F_2^k \longrightarrow Gx \in F_2^n$	$y \in C \longrightarrow Hy = \mathbf{0} \in F_2^{n-k}$

$$HG = \mathbf{0}$$

$ker(G^T)$	$ker(H)$
H	G^T
$n-k$	k

$$y = Gx \qquad y' = y + e$$

$$Hy' = HGx + He = He$$

He : Error syndrome

F_2

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Dual code

$$C \longleftrightarrow C^{\perp}$$

$$\boxed{\begin{array}{c} G \longleftrightarrow H^T \\ H \longleftrightarrow G^T \end{array}}$$

$$C^{\perp} = \{x \in F_2^n \,|\, x \cdot c = 0 \text{ for all } c \in C\}$$

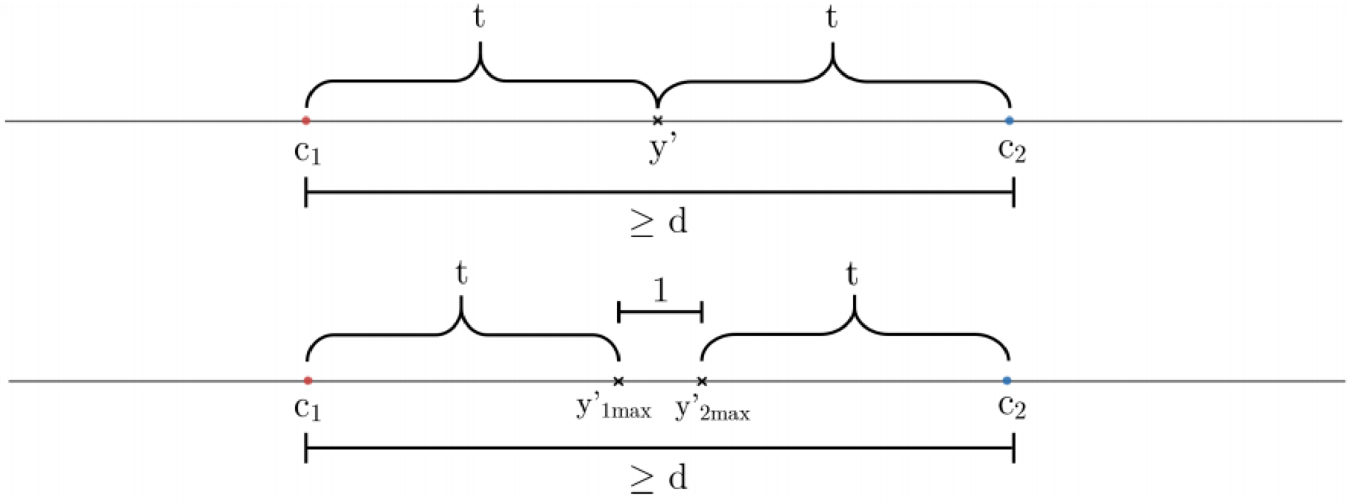
$$dim(C) + dim(C^{\perp}) = n$$

Weakly self-dual code	**Self-dual code**
$C \subset C^{\perp}$	$C = C^{\perp}$

Example

$$d((1\,0\,1),(0\,0\,1)) = 1$$

A code with distance d corrects errors on t qubits: $d \geq 2t+1$



$$d(C) \equiv \min_{x,y\in C, x\neq y} d(x,y) \qquad wt(x) \equiv d(x,0)$$

The distance d of a linear code C equals the minimum number of linearly dependent columns of the parity check matrix H.

$$d-1 \leq n-k$$

Quantum Error Correction

$$|s_{k-1}\dots s_0\rangle \longrightarrow |(s_{k-1}\dots s_0)_L\rangle$$

k qubit state

n qubit state

The code lives in a 2^k dimensional subspace

$$[n,k,d]$$

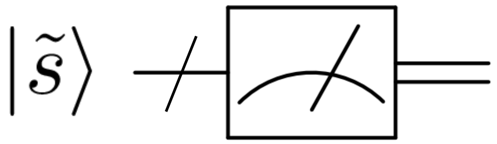
Code with distance d which encodes k qubits of information using n qubits

Example

$$|0_L\rangle = |000\rangle$$

$$|1_L\rangle = |111\rangle$$

$$|\bar{s}\rangle \xrightarrow{\text{Noise}} |\tilde{s}\rangle \xrightarrow{\text{Error-correction}} |\bar{s}\rangle$$

Error detection	**Recovery**
 Error syndrome	Error syndrome $\longrightarrow$ $U_{es}\,|\tilde{s}\rangle = |\bar{s}\rangle$

Error-correction

- May be separated into two steps: the error-detection or syndrome diagnosis and the recovery.

$$E_i = e_{i0}I + e_{i1}X + e_{i2}Z + e_{i3}XZ$$
$$\{I, X, Z, ZX\}^{\otimes n}$$

An error-correction code can correct some set of errors iff it satisfies the quantum error-correction conditions, and any set of errors whose errors are linear combinations of the errors from the correctable set is also correctable.

$$\langle l| E_i^\dagger E_j |k\rangle = \alpha_{ij}\delta_{lk}$$
$$PE_i^\dagger E_j P = \alpha_{ij} P$$

$$F_j = \sum_i m_{ji} E_i$$

Degenerate Code	**Non-degenerate Code**
$\alpha_{ij} \neq \delta_{ij}$	$\alpha_{ij} = \delta_{ij}$

Stabilizer Codes

Definition 1.1 *Let $S = \{M_1, \dots, M_m\} \subseteq \mathcal{P}_n$ be a group (with regular matrix multiplication as group operation) of operators, then the nontrivial subspace which is invariant to the operators $\{M_i\}$, i.e. $M_i P = P$ (P being the projector onto that subspace), is said to be stabilized by the stabilizer S.*

$$\mathcal{P}_n = \{\pm i, \pm 1\} \times \{I, X, Z, Y\}^{\otimes n}$$

$$[M_i, M_j] = 0$$

$$-I, \pm iI \notin S$$

$$M = M^\dagger$$

$$\Downarrow$$

$$\mathcal{P}'_n = \pm 1 \times \{I, X, Z, iXZ\}^{\otimes n} \subset \mathcal{P}_n$$

$$S = \langle \hat{M}_1, \dots, \hat{M}_l \rangle$$

Proposition 1.1
If G is a group (with identity e), and $H \subset G$ is non-empty, then H is a subgroup of G if and only if $hh'^{-1} \in H$, for any $h, h' \in H$.

$$S' = \langle \hat{M}_1, \dots, \hat{M}_{l-1} \rangle$$

$$\mathcal{P}'_n = \pm 1 \times \{I, X, Z, Y\}^{\otimes n} = \pm 1 \times \{I, X, Z, iXZ\}^{\otimes n}$$
$$= \pm i^y \times \{I, X, Z, XZ\}^{\otimes n}$$

$$\boxed{s = (s_1, s_2) \longleftrightarrow M \in \mathcal{P}_n}$$

$$f : \{0,1\}^{2n} \to \mathcal{P}'_n$$
$$f(s_1, s_2) = \pm i^{s_1^T s_2} \bigotimes_{j=1}^{n} X^{s_1^j} Z^{s_2^j}$$

Commutativity

$$f(s_1, s_2)f(s'_1, s'_2) = (-1)^{s_2^T s'_1 + s_1^T s'_2} f(s'_1, s'_2)f(s_1, s_2)$$

Commutativity $\Leftrightarrow \; s_2^T s'_1 + s_1^T s'_2 \equiv_2 0 \; \Leftrightarrow \; s^T \Lambda s' \equiv_2 0$

Unitarity and Hermiticity

$$M = M^\dagger \implies (MM^\dagger = I \Leftrightarrow M^2 = I)$$

$$f(s_1, s_2)^2 = (-1)^{s_1^T s_2} I$$

Unitarity and Hermiticity $\Leftrightarrow \; y = s_1^T s_2 \equiv_2 0$

$$s = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \quad \Bigg| \quad s^T \Lambda s' = 0, \; \Lambda = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad \Bigg| \quad s' = \begin{bmatrix} s'_1 \\ s'_2 \end{bmatrix}$$

Independent set of generators

$$G = \{g_i\} \quad \forall g \in G,\ \nexists k_j \in \mathbb{N} \wedge \nexists g_j \in G \setminus \{g\} : g = \prod_j g_j^{k_j}$$

$$H = \begin{bmatrix} s_1^T \\ \vdots \\ s_r^T \end{bmatrix}$$

$$\prod_l f(s_l)^{k_l} = \alpha(s_1, \ldots, s_r, k_1, \ldots, k_r) f\Big(\sum_l k_l s_l\Big)$$

$$f(s_g) \neq \alpha(\ldots) f\Big(\sum_{g' \in G \setminus \{g\}} k_{g'} s_{g'}\Big) \Leftrightarrow s_g \neq \sum_{g' \in G \setminus \{g\}} k_{g'} s_{g'}$$

Independency of generators $\Leftrightarrow$ Linear independency of corresponding binary strings

Theorem 1.1 *Let S be the stabilizer for a stabilizer code $C(S)$. Suppose $\{E_j\}$ is a set of operators in $\mathcal{P}_n$ such that $E_j^\dagger E_k \notin Z(S) - S$ for all j and k. Then $\{E_j\}$ is a correctable set of errors for the code $C(S)$.*

$$S = \{M_1, \ldots, M_m\}$$
$$S = \langle \hat{M}_1, \ldots, \hat{M}_l \rangle$$

$$M_i P = P \longrightarrow P = \sum_i |c_i\rangle \langle c_i| \longrightarrow \{|c_i\rangle\}$$

Correction of errors

Commutes with all generators	Anti-commutes with at least one generator
$E \in S$ ✔ $E \in Z(S) - S$ ✖	$\hat{M} E \|c_i\rangle = -E\hat{M} \|c_i\rangle = -E \|c_i\rangle$ $E \notin Z(S)$ ✔

$$Z(S) = \{E \in \mathcal{P}_n : EM = ME \text{ for all } M \in S\} \overset{*}{=} \{E \in \mathcal{P}_n : ES = SE\} = N(S)$$

$*$ Centralizer of S coincides with the normalizer of S

Degenerate code	Non-degenerate code
$E_j P E_j = E_{j'} P E_{j'}$	$E_j^\dagger E_k \in \mathcal{P}_n - Z(S), \forall j,k : j \neq k$ $E_j^\dagger E_j = I \in S$
E_j corrects $E_{j'}$ $E_{j'}$ corrects E_j	$P E_i^\dagger E_j P = \alpha_{ij} P$ $\alpha_{ij} = \delta_{ij}$
$P E_i^\dagger E_j P = \alpha_{ij} P$ $\alpha_{ij} \neq \delta_{ij}$	

		Distance
Weight	The weight of an operator belonging to Pn is the number of terms in the tensor product which are not equal to the identity, and in the stabilizer binary string notation introduced $w(s) = \sum_j s_x^j \vee s_z^j \qquad s = \begin{bmatrix} s_x \\ s_z \end{bmatrix}$	The distance of a stabilizer code C(S) is defined to be the minimum weight of an element of N(S)−S

Examples of Stabilizer Codes

Definition 2.1 Let C_1 and C_2 be $[n, k_1]$ and $[n, k_2]$ classical linear codes such that $C_2 \subset C_1$, and C_1 and $C_2^\perp$ both correct t errors. An $[n, k_1 - k_2]$ quantum code, $CSS(C_1, C_2)$, capable of correcting errors on up to t qubits can be created and is said to be a CSS code of C_1 over C_2. Moreover, letting the classical codeword $x_i \in C_1$ be such that $x_i - x_j \notin C_2$ for $i \neq j$ one has

$$|x_i + C_2\rangle \equiv \frac{1}{\sqrt{|C_2|}} \sum_{y \in C_2} |x_i + y\rangle,$$

with the $\{|x_i + C_2\rangle\}$ spanning the code subspace.

C_1

$x_j + C_2$

$\vdots$

$x_i + C_2$

$x_i - x_j \notin C_2$

$\uparrow$

$aH = bH \Leftrightarrow a^{-1}b \in H$

Dimension of the code subspace

$|C_1|/|C_2| = 2^{k_1 - k_2}$

$$\boxed{\begin{gathered} HZH = X, \ HXH = Z \\ C_2^\perp \overset{H}{\leftrightarrow} C_2 \end{gathered}}$$

$\sum_{y \in C} (-1)^{x \cdot y}$	
$x \in C^\perp$	$x \notin C^\perp$
$\sum_{y \in C} (-1)^{x \cdot y} = \lvert C\rvert$	$\sum_{y \in C} (-1)^{x \cdot y} = 0$

Error correction

$$|x_i + C_2\rangle = \frac{1}{\sqrt{|C_2|}} \sum_{y\in C_2} |x_i + y\rangle \overset{H^{\otimes n}}{\rightarrow} \frac{1}{\sqrt{|C_2|2^n}} \sum_{z} \sum_{y\in C_2} (-1)^{z\cdot(x_i+y)} |z\rangle = \sqrt{\frac{|C_2|}{2^n}} \sum_{z\in C_2^{\perp}} (-1)^{z\cdot x_i} |z\rangle$$

$$\frac{1}{\sqrt{|C_2|}} \sum_{y\in C_2} (-1)^{e_2\cdot(x_i+y)} |x_i + y + e_1\rangle |0\rangle \rightarrow \frac{1}{\sqrt{|C_2|}} \sum_{y\in C_2} (-1)^{e_2\cdot(x_i+y)} |x_i + y + e_1\rangle |H_1 e_1\rangle$$

Bit flips

$$\frac{1}{\sqrt{|C_2|}} \sum_{y\in C_2} (-1)^{e_2\cdot(x_i+y)} |x_i + y\rangle \rightarrow \frac{1}{\sqrt{|C_2|2^n}} \sum_{z} \sum_{y\in C_2} (-1)^{(z+e_2)\cdot(x_i+y)} |z\rangle$$

$$= \frac{1}{\sqrt{|C_2|2^n}} \sum_{z'} \sum_{y\in C_2} (-1)^{z'\cdot(x_i+y)} |z' + e_2\rangle$$

$$= \sqrt{\frac{|C_2|}{2^n}} \sum_{z'\in C^{\perp}} (-1)^{z'\cdot x_i} |z' + e_2\rangle ,$$

Phase flips

$z' \equiv z + e_2$

$$\sqrt{\frac{|C_2|}{2^n}} \sum_{z'\in C^{\perp}} (-1)^{z'\cdot x_i} |z'\rangle \overset{H^{\otimes n}}{\rightarrow} \frac{1}{\sqrt{|C_2|}} \sum_{y\in C_2} |x_i + y\rangle$$

Error correction

$$\frac{1}{\sqrt{|C_2|}} \sum_{y \in C_2} (-1)^{e_2 \cdot (x_i + y)} |x_i + y + e_1\rangle$$

Bit flips

$$\frac{1}{\sqrt{|C_2|}} \sum_{y \in C_2} (-1)^{e_2 \cdot (x_i + y)} |x_i + y + e_1\rangle |0\rangle \rightarrow \frac{1}{\sqrt{|C_2|}} \sum_{y \in C_2} (-1)^{e_2 \cdot (x_i + y)} |x_i + y + e_1\rangle |H_1 e_1\rangle$$

Phase flips

$z' \equiv z + e_2$

$$\begin{aligned}\frac{1}{\sqrt{|C_2|}} \sum_{y \in C_2} (-1)^{e_2 \cdot (x_i + y)} |x_i + y\rangle &\rightarrow \frac{1}{\sqrt{|C_2| 2^n}} \sum_{z} \sum_{y \in C_2} (-1)^{(z + e_2) \cdot (x_i + y)} |z\rangle \\ &= \frac{1}{\sqrt{|C_2| 2^n}} \sum_{z'} \sum_{y \in C_2} (-1)^{z' \cdot (x_i + y)} |z' + e_2\rangle \\ &= \sqrt{\frac{|C_2|}{2^n}} \sum_{z' \in C^\perp} (-1)^{z' \cdot x_i} |z' + e_2\rangle ,\end{aligned}$$

$$\sqrt{\frac{|C_2|}{2^n}} \sum_{z' \in C^\perp} (-1)^{z' \cdot x_i} |z'\rangle \overset{H^{\otimes n}}{\rightarrow} \frac{1}{\sqrt{|C_2|}} \sum_{y \in C_2} |x_i + y\rangle$$

The code $CSS(C_1, C_2)$ is an $[n, k_1 - k_2, d \geq min(d_1, d_2^\perp)]$ quantum code (with $d = min(d_1, d_2^\perp)$ if the code is non-degenerate), with d_1 and d_2 relating to $t_1 = \lfloor \frac{d_1 - 1}{2} \rfloor$ and $t_2 = \lfloor \frac{d_2 - 1}{2} \rfloor$, and correcting errors occurring in at least up to $min(t_1, t_2)$ qubits.

Check matrix

$$\begin{bmatrix} H(C_2^\perp) & \mathbf{0} \\ \mathbf{0} & H(C_1) \end{bmatrix}$$

Commutativity $\Leftrightarrow$ $H(C_2^\perp)H(C_1)^T = 0$

$$H(C_2^\perp)H(C_1)^T = [H(C_1)G(C_2)]^T \underset{\substack{\uparrow \\ C_2 \subset C_1}}{=} 0$$

<u>Bit flip errors</u>

<u>Phase flip errors</u>

$$\underset{C_1}{X} \xleftrightarrow{H(\cdot)H} \underset{C_2^\perp}{Z}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow |\bar{\psi}\rangle = \alpha|0_L\rangle + \beta|1_L\rangle$$

The Shor code is a $[9,1,3]$ degenerate CSS code

$$|0_L\rangle \equiv \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}} \qquad |0_L\rangle \equiv \frac{(|0'\rangle + |1'\rangle)(|0'\rangle + |1'\rangle)(|0'\rangle + |1'\rangle)}{2\sqrt{2}}$$

$$|1_L\rangle \equiv \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}} \qquad |1_L\rangle \equiv \frac{(|0'\rangle - |1'\rangle)(|0'\rangle - |1'\rangle)(|0'\rangle - |1'\rangle)}{2\sqrt{2}}$$

$$|0'\rangle = |000\rangle \quad |1'\rangle = |111\rangle$$

$$(|a_0\rangle + e^{i\pi n_0}|\bar{a}_0\rangle)$$

$$\downarrow Z_1Z_2 \qquad\qquad \downarrow Z_2Z_3$$

$$(e^{i\pi b_1}e^{i\pi b_2}|a_0\rangle + e^{i\pi n_0}e^{i\pi\bar{b}_1}e^{i\pi\bar{b}_2}|\bar{a}_0\rangle) \qquad (e^{i\pi b_2}e^{i\pi b_3}|a_0\rangle + e^{i\pi n_0}e^{i\pi\bar{b}_2}e^{i\pi\bar{b}_3}|\bar{a}_0\rangle)$$

$$\langle Z_1Z_2\rangle = e^{i\pi b_1}e^{i\pi b_2} \qquad \langle Z_2Z_3\rangle = e^{i\pi b_2}e^{i\pi b_3}$$

$$|0''_L\rangle = |+'\rangle\,|+'\rangle\,|+'\rangle \qquad |1''_L\rangle = |-'\rangle\,|-'\rangle\,|-'\rangle$$

$$\boxed{\begin{array}{c|c|c} (|a_0\rangle + e^{i\pi n_0}\,|\bar{a}_0\rangle) & (|a_1\rangle + e^{i\pi n_1}\,|\bar{a}_1\rangle) & (|a_2\rangle + e^{i\pi n_2}\,|\bar{a}_2\rangle) \\ \downarrow X_1X_2X_3 & \downarrow X_4X_5X_6 & \downarrow X_7X_8X_9 \\ (|\bar{a}_0\rangle + e^{i\pi n_0}\,|a_0\rangle) & (|\bar{a}_1\rangle + e^{i\pi n_1}\,|a_1\rangle) & (|\bar{a}_2\rangle + e^{i\pi n_2}\,|a_2\rangle) \end{array}} \quad n_0, n_1, n_2 \in \{0, 1\}$$

$$\langle X_1X_2X_3\rangle = e^{i\pi n_0} \qquad \langle X_4X_5X_6\rangle = e^{i\pi n_1} \qquad \langle X_7X_8X_9\rangle = e^{i\pi n_2}$$

$$X_1X_2X_3X_4X_5X_6 \quad X_4X_5X_6X_7X_8X_9$$

Name	Operator	Check matrix	Generator matrices
g_1	$ZZIIIIIII$	$\left[\begin{array}{cc} H(C_2^\perp) = \begin{matrix} 111111000 \\ 000111111 \end{matrix} & \mathbf{0} \\ \mathbf{0} & H(C_1) = \begin{matrix} 110000000 \\ 011000000 \\ 000110000 \\ 000011000 \\ 000000110 \\ 000000011 \end{matrix} \end{array}\right]$	$G(C_1) = \begin{bmatrix} 111000000 \\ 000111000 \\ 000000111 \end{bmatrix}^T$ $G(C_2^\perp) = \begin{bmatrix} 110000000 \\ 011000000 \\ 000110000 \\ 000011000 \\ 000000110 \\ 000000011 \\ 100100100 \end{bmatrix}^T$
g_2	$IZZIIIIII$		
g_3	$IIIZZIIII$		
g_4	$IIIIZZIII$		
g_5	$IIIIIIZZI$		
g_6	$IIIIIIIZZ$		
g_7	$XXXXXXIII$		
g_8	$IIIXXXXXX$		
$\bar{Z}$	$XXXXXXXXX$		
$\bar{X}$	$ZZZZZZZZZ$		

$$H(C_1)G(C_2) = H(C_1)H(C_2^\perp)^T = \mathbf{0}$$

$$C_2 \subseteq C_1$$

Steane Code

The Steane code is an example of a CSS code which uses the classical $[7,4,3]$ Hamming code C with $C_1 \equiv C$ and $C_2 \equiv C^{\perp}$, and associated matrices

$$H(C_1) = G(C_2)^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad H(C_2) = G(C_1)^T = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$H(C_1)G(C_2) = H(C_1)H(C_1)^T = \mathbf{0} \implies C_2 \subseteq C_1$$

$$C_2^{\perp} = C = C_1$$

$$\begin{bmatrix} H(C) & \mathbf{0} \\ \mathbf{0} & H(C) \end{bmatrix}$$

C_2	C_1	$CSS(C_1, C_2)$
$[7, 7-4, d=4 \geq 3] = [7,3,4]$	$[7,4,3]$	$[7, 4-3, 3 \geq min(3,4)] = [7,1,3]$

C_1

$\boxed{C_2}$

$\boxed{C_2 + [1110000]}$

$$|0_L\rangle = \prod_{i=1}^{n-k} \frac{(I+g_i)}{2^{n-k}} \frac{|0000000\rangle}{\sqrt{|C_2|/2^{n-k}}} = \frac{1}{\sqrt{|C_2| = 8}} \sum_{\text{even } s \in C} |s\rangle = |0000000\rangle + |1010101\rangle + |0110011\rangle$$
$$+ |0001111\rangle + |1100110\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle$$

$$|1_L\rangle = \prod_{i=1}^{n-k} \frac{(I+g_i)}{2^{n-k}} \frac{|1111111\rangle}{\sqrt{|C_2|/2^{n-k}}} = \frac{1}{\sqrt{|C_2| = 8}} \sum_{\text{odd } s \in C} |s\rangle = |1111111\rangle + |0100101\rangle + |1000011\rangle$$
$$+ |1110000\rangle + |0010110\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle$$

The stabilizer codes are a simple class of quantum error correction codes with a remarkable similarity to classical linear codes. However, unlike classical error correction codes, quantum error correction codes may have that codewords are sent to non-orthogonal subspaces for some errors, with the corrupted codeword being correctable (for any of the errors on the correctable set). Such codes are called degenerate codes.